

TWO STATEMENTS OF THE DUGGAN-SCHWARTZ THEOREM

EGOR IANOVSKI

ABSTRACT. The Duggan-Schwartz theorem [Duggan and Schwartz, 1992] is a famous result concerning strategy-proof social choice correspondences, often stated as “A social choice correspondence that can be manipulated by neither an optimist nor a pessimist has a weak dictator”. However, this formulation is actually due to Taylor [2002], and the original theorem, at face value, looks rather different. In this note we show that the two are in fact equivalent.

1. DEFINITIONS

Definition 1.1. Let V be a finite set of voters, A a finite set of alternatives.

A profile P consists of a linear order over A (also known as a *preference order* or a *ballot*), P_i , for every voter i . The set of all profiles of voters V over alternatives A is denoted $\mathcal{P}(V, A)$. We use P_{-i} to refer to the ballots of all voters except i . Hence, $P = P_i P_{-i}$ and $P'_i P_{-i}$ is obtained from profile P by replacing P_i with P'_i .

A *social choice correspondence* produces a nonempty set of alternatives, $F : \mathcal{P}(V, A) \rightarrow 2^A \setminus \{\emptyset\}$.

Definition 1.2. Let $\emptyset \neq W \subseteq A$. We use $\text{best}(P_i, W)$ to denote the best alternative in W according to P_i , $\text{worst}(P_i, W)$ the worst.

We extend \geq_i into two weak orders over $2^A \setminus \{\emptyset\}$:

- (1) $X \geq_i^O Y$ iff $\text{best}(P_i, X) \geq_i \text{best}(P_i, Y)$.
- (2) $X \geq_i^P Y$ iff $\text{worst}(P_i, X) \geq_i \text{worst}(P_i, Y)$.

A social choice correspondence is *strategy-proof for optimists* (SPO) if for all P'_i , whenever $F(P_i P_{-i}) = W$ and $F(P'_i P_{-i}) = W'$, $W \geq_i^O W'$. A social choice correspondence is *strategy-proof for pessimists* (SPP) if for all P'_i , whenever $F(P_i P_{-i}) = W$ and $F(P'_i P_{-i}) = W'$, $W \geq_i^P W'$.

Definition 1.3. Given a social choice correspondence F , a *weak dictator* is some $i \in V$ such that the first choice of i is always in $F(P)$.

2. PROOFS

Theorem 2.1 (Taylor [2002]). *Let F be a social choice correspondence that satisfies SPP, SPO and is onto with respect to singletons. That is, for every $a \in A$ there exists a P such that $F(P) = \{a\}$.*

For $|A| \geq 3$, F has a weak dictator.

Theorem 2.2 (Duggan and Schwartz [1992]). *Let F be a social choice correspondence that is onto with respect to singletons. That is, for every $a \in A$ there exists a P such that $F(P) = \{a\}$.*

Let each voter i be equipped with a probability function $p_i : \mathcal{P}(V, A) \times A \times 2^A \rightarrow [0, 1]$ such that $\sum_{x \in X} p_i(P, x, X) = 1$ and $p_i(P, a, X) > 0$ whenever $a = \text{best}(P_i, X)$ or $a = \text{worst}(P_i, X)$.

Suppose further that for every u_i consistent with P_i ($u_i(a) > u_i(b)$ whenever $a >_i b$), and for every P'_i , the following is true:

$$\sum_{x \in F(P_i P_{-i})} p_i(P_i P_{-i}, x, F(P_i P_{-i})) u_i(x) \geq \sum_{x \in F(P'_i P_{-i})} p_i(P_i P_{-i}, x, F(P'_i P_{-i})) u_i(x).$$

For $|A| \geq 3$, F has a weak dictator.

The notion of manipulation used by Duggan and Schwartz is obviously more general than that of Taylor, and one is thus tempted to conclude that the original theorem is weaker than Taylor's reformulation.¹

However, this would be erroneous as the theorems, strictly speaking, are incomparable. Taylor's theorem concerns a social choice correspondence F , whereas Duggan and Schwartz's theorem applies to F together with a set of probability functions, p_i . It is entirely plausible that one could find two sets of probability functions such that F and p_1, \dots, p_n satisfy the hypotheses of the Duggan-Schwartz theorem while F and p'_1, \dots, p'_n do not. However, F is unchanged – it either has a weak dictator, or it does not.

To more properly compare the two theorems, then, we need to take an existential projection over the original Duggan-Schwartz theorem.

Theorem 2.3 (Duggan and Schwartz [1992]). *Let F be a social choice correspondence that is onto with respect to singletons. That is, for every $a \in A$ there exists a P such that $F(P) = \{a\}$.*

Suppose there exist probability functions $p_i : \mathcal{P}(V, A) \times A \times 2^A \rightarrow [0, 1]$ such that $\sum_{x \in X} p_i(P, x, X) = 1$ and $p_i(P, a, X) > 0$ whenever $a = \text{best}(P_i, X)$ or $a = \text{worst}(P_i, X)$.

Suppose further that for every u_i consistent with P_i ($u_i(a) > u_i(b)$ whenever $a \succ_i b$), and for every P'_i , the following is true:

$$\sum_{x \in F(P_i P_{-i})} p_i(P_i P_{-i}, x, F(P_i P_{-i})) u_i(x) \geq \sum_{x \in F(P'_i P_{-i})} p_i(P_i P_{-i}, x, F(P'_i P_{-i})) u_i(x).$$

For $|A| \geq 3$, F has a weak dictator.

Now we claim the two theorems are equivalent.

Proposition 2.4. *F satisfies the hypotheses of Theorem 2.1 if and only if F satisfies the hypotheses of Theorem 2.3.*

Proof. We will first show that if F is manipulable in the sense of Taylor it is manipulable in the sense of Duggan-Schwartz. Pay heed to the order of the quantifiers in Theorem 2.3, as they may appear counter-intuitive: F is strategy-proof if for *some* choice of probability functions, for *every* choice of a utility function, voter i cannot improve his expected utility. Hence, F is manipulable just if for *every* choice of probability functions we can construct *some* utility function giving voter i a profitable deviation.

Suppose i can manipulate optimistically from $P_i P_{-i}$ to $P'_i P_{-i}$. That is:

$$\begin{aligned} F(P_i P_{-i}) &= X, & F(P'_i P_{-i}) &= Y, \\ \text{best}(P_i, X) &= a, & \text{best}(P_i, Y) &= b, \\ a &<_i b. \end{aligned}$$

Now, let p_i be any probability function in the sense of Theorem 2.3. Note that this means that $p_i(P, a, X) = \epsilon$ and $p_i(P, b, Y) = \delta$ are strictly positive. Let $c \in X$ be the next-best alternative after a . Observe that an upper bound on the utility voter i obtains sincerely is $\epsilon u_i(a) + (1 - \epsilon) u_i(c)$, whereas the lower bound on the utility voter i obtains from the deviation is $\delta u_i(b)$. All we need to do is pick a u_i that satisfies:

$$\delta u_i(b) > \epsilon u_i(a) + (1 - \epsilon) u_i(c).$$

It is of course easy to do so as, necessarily, $u_i(b) > u_i(a), u_i(c)$, and ϵ, δ are constants. For example, let $u_i(a) = 1, u_i(c) = 2$ and $u_i(b) = 3/\delta$.

¹A wider notion of manipulability implies a more narrow notion of strategy-proofness, and hence the theorem would apply to less functions.

Suppose i can manipulate pessimistically from $P_i P_{-i}$ to $P'_i P_{-i}$. That is:

$$\begin{aligned} F(P_i P_{-i}) &= X, & F(P'_i P_{-i}) &= Y, \\ \text{worst}(P_i, X) &= a, & \text{worst}(P_i, Y) &= b, \\ a &<_i b. \end{aligned}$$

As before, let p_i be any probability function in the sense of Theorem 2.3. This means that $p_i(P, a, X) = \epsilon$ and $p_i(P, b, Y) = \delta$ are strictly positive. Let $c \in X$ be the best alternative in the set. Observe that an upper bound on the utility voter i obtains sincerely is $\epsilon u_i(a) + (1 - \epsilon)u_i(c)$, whereas the lower bound on the utility voter i obtains from the deviation is $u_i(b)$.² All we need to do is pick a u_i that satisfies:

$$u_i(b) > \epsilon u_i(a) + (1 - \epsilon)u_i(c).$$

This time it is possible that $u_i(c) > u_i(b)$, however $1 - \epsilon$ is strictly smaller than 1. One possibility is $u_i(a) = 1$, $u_i(b) = \frac{1/\epsilon + \epsilon + 1}{1 - \epsilon}$, $u_i(c) = \frac{1/\epsilon + \epsilon + 2}{1 - \epsilon}$. This leads to the following inequality, which can be verified algebraically:

$$\frac{1/\epsilon + \epsilon + 1}{1 - \epsilon} > 2\epsilon + 2 + 1/\epsilon.$$

Now suppose that F is manipulable in the sense of Duggan-Schwartz. This means for every choice of p_i , there is some choice of u_i such that for some choice of $P_i P_{-i}$ and $P'_i P_{-i}$, i 's expected utility is higher in the insincere profile.

Pick a p_i that attaches a probability of $1/2$ to the best alternative in the set and $1/2$ to the worst. In other words, we have the following situation:

$$\begin{aligned} F(P_i P_{-i}) &= X, & F(P'_i P_{-i}) &= Y, \\ \text{best}(P_i, X) &= x_1, & \text{best}(P_i, Y) &= y_1, \\ \text{worst}(P_i, X) &= x_2, & \text{worst}(P_i, Y) &= y_2, \\ \frac{u_i(x_1) + u_i(x_2)}{2} &< \frac{u_i(y_1) + u_i(y_2)}{2}. \end{aligned}$$

Clearly, a necessary condition for the above to hold is that either $u_i(y_1) > u_i(x_1)$ or $u_i(y_2) > u_i(x_2)$. That is to say, F is manipulable by either an optimist or a pessimist. \square

REFERENCES

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- Alan D. Taylor. The manipulability of voting systems. *The American Mathematical Monthly*, 109 (4):321–337, 2002. ISSN 00029890, 19300972. URL <http://www.jstor.org/stable/2695497>.

² b is the worst element in Y , so the utility of any other element must be at least $u_i(b)$, and $p_i(P, y, Y)$ sums to 1.